Problem VI.3 ... repellent resistive bipyramids

5 points; průměr 2,48; řešilo 46 studentů

We have a model of a regular N-gonal bipyramid made of conducting wires. The connections in the plane of symmetry each have resistance R_2, whereas the connections going from one of the vertices to a point in the base have resistance R_1. Determine the resistance between

1. *main vertices (above and below the base plane),*
2. *adjacent vertices in the base plane,*
3. *opposite vertices in the base plane (the most distant ones) for even values of N.*

Karel wanted N-gonal bipyramids.

All subtasks can be solved in many ways. In general, we will be looking for symmetries that will help us solve them. Some branches in a circuit can be removed in our calculations immediately because no current flows through them. Another symmetry will help us by letting us only calculate the resistance of some parts of a circuit. To view things more easily, let's assume that the plane of the common base of both pyramids (let's call it the main plane) is horizontal and the remaining two vertices of the pyramids are lying one above and one below the main plane.

Between the main vertices

If we attach the voltage source to the main vertices, all the resistive wires in the main plane have the same potential. That is because there is the same ratio of resistances above and below the main plane for each of the vertices in this plane. If there is no potential difference between vertices in a branch, there is no current flowing through this branch, and we can remove it from the circuit and simplify the scheme. In the figure 2, we can see that the situation simplifies to N branches connected in parallel and each branch containing only two wires with resistance R_1. The total resistance of this circuit is

$$\frac{1}{R_{aN}} = \sum_{i=1}^{N} \frac{1}{2R_1} \implies R_{aN} = \frac{2R_1}{N}\,.$$

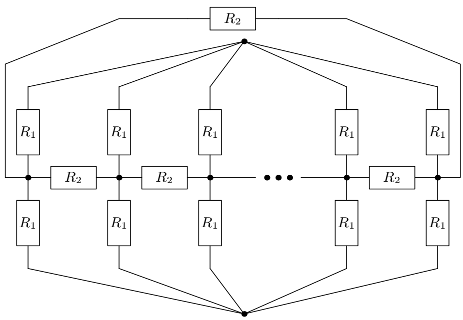

Fig. 1: Diagram for wiring across adjacent base vertices.

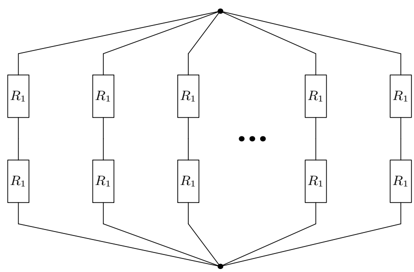

Fig. 2: Simplified diagram for wiring a bipyramid through the main vertices; the number of vertical branches is N.

Between adjacent vertices in the main plane

The next two subtasks (most likely) cannot generally be solved as simply as subtask a), but with some imagination, we can find an algorithm that lets us solve them for any given N. As a bonus, we will show how to write down this algorithm using recurrence relations which will make it easier to find a general solution for $N + k$.

even N In this case, the object is symmetrical with respect to the plane which is perpendicular to the line segment connecting the two adjacent vertices attached to the voltage source and lies in the middle of this segment. Let's call it the vertical plane of symmetry. The line segments that run from the main plane to the upper main vertex can be divided into pairs, that are symmetrical with respect to it. For each pair, current flows into the upper main vertex through one of these wires and flows out through the other wire, and the magnitudes of the currents in these wires must be the same due to symmetry. Therefore, we may assume that in the upper main vertex, these pairs of wires from different vertices of the N-gon are not connected with other pairs. That does not affect the resulting resistance of this circuit. The same idea applies to the half-space below the main plane.

We will now introduce one more labeling definition. The object's orientation shall be such that the line segment connected to the voltage source is called the front segment. The line segment on the opposite side of the N-gon is called the back segment. We can find the answer to the problem by moving from the back to the front.

Let's start at the back segment. From each of its endpoints, two wires lead to the main vertices, one above and one below. As we mentioned, we can treat the connections of these wires in the main vertices as separate thanks to symmetries. The net resistance R_z between the endpoints of the back segment then satisfies

$$\frac{1}{R_z} = \frac{1}{R_2} + \frac{1}{2R_1} + \frac{1}{2R_1}\,,$$

since we are dealing with a parallel connection of three circuits with resistances R_2 (the back segment), $2R_1$ (the path from one endpoint of the back segment to the other endpoint through the upper main vertex), and $2R_1$ again (similarly through the lower main vertex).

Next, we move toward the front segment. In the main plane, two wires with resistances R_2 lead from the back segment, and at the ends of these wires, there are vertices from which wires lead to main vertices again. We want to calculate the resistance of this extended "back" part of the circuit if a voltage source is attached to its endpoints (the two new vertices). Just like before, we can separate this part from the rest of the circuit in the main vertices. We have a parallel connection of three branches here again. Two of these branches have resistances $2R_1$, and the last one, which lies in the main plane, consists of a circuit with resistance R_z and two wires with resistances R_2 connected in series. If we denote the net resistance between these vertices as R_{z3} (3 in the index means that we have accounted for 3 sides of the N-gon), then

$$\frac{1}{R_{z3}} = \frac{1}{R_2 + R_2 + R_z} + \frac{1}{2R_1} + \frac{1}{2R_1}\,.$$

We can proceed further to the front in the same way, obtaining net resistances $R_{z2}, R_{3z3}, \ldots$, until we calculate $R_{z(N-1)}$. We only need to connect the resistance R_2 on the front segment to this circuit in parallel, and we have prevailed.

$$\frac{1}{R_{bN}} = \frac{1}{R_{z(N-1)}} + \frac{1}{R_2}$$

Let's move on to recurrence. We assume that $R_{b(N-2)}$ is known and we want to obtain R_{bN}. We need to subtract R_2 in parallel from $R_{b(N-2)}$ (obtaining $R_{z(N-3)}$), add R_2 twice in series, then add $2R_1$ twice in parallel to the whole thing (obtaining $R_{z(N-1)}$) and finally add R_2 in parallel. Written as an equation, it is

$$\frac{1}{R_{bN}} = \frac{1}{R_2} + \frac{1}{R_1} + \frac{1}{2R_2 + \left(\frac{1}{R_{b(N-2)}} - \frac{1}{R_2}\right)^{-1}}\,.$$

We will introduce the notation

$$G_{bK} = \frac{R_1}{R_{bK}}$$

and for clarity denote the ratio

$$p = \frac{R_1}{R_2}\,.$$

Then, we can write the recurrence relation as

$$R_1 \frac{1}{R_{bN}} = 1 + p + p\frac{G_{bN-2} - p}{2G_{bN-2} - p}\,.$$

All that remains is finding the initial condition. The lowest possible $N-2$ is 4, since we are dealing with a pyramid. That gives us

$$G_{b4} = 1 + p + \frac{p\,(p+1)}{3p+2}\,.$$

Since we now know the recurrence and its initial condition, we can use the program Mathematica to find the (partial) general solution for even N

$$R_{bN} = R_2\left(\frac{2}{\sqrt{2p+1}\left(1 - \left(\frac{-p+\sqrt{2p+1}-1}{p^2}\right)^{-N/2}\left(-\frac{p+\sqrt{2p+1}+1}{p^2}\right)^{N/2}\right)} - \frac{1}{\sqrt{2p+1}} + 1\right)\,.$$

Now we get to the case in which N is odd.

odd N Once again, let's assume that the object is oriented in such a way that voltage is attached to the front segment. Now we do not have a back segment, but a back point, from which resistive wires lead to the main vertices. Due to symmetry, we can realize that current does not flow through these wires (!) That is because the same current must flow into and out of the "back point" in the main plane, and at the same time, current cannot flow into it from one of the main vertices and out of it into the other main vertex, because that would contradict symmetry.

Therefore, we do not need to consider wires from the back vertex to the main vertices. Other than that, we will proceed in the same way as for even N, the only difference being that the resistance on the "back segment" is now $2R_2$ (the back segment is now basically between those furthermost vertices in the main plane from which current flows into the main vertices or vice versa).

We will use the recurrence relation from the previous part again. The smallest odd N for which the problem makes sense is $N = 3$. From the text above, we can deduce that the resistance of such an object is

$$R_{b3} = \left(\frac{1}{R_2} + \frac{1}{2R_2} + \frac{1}{2R_1} + \frac{1}{2R_1}\right)^{-1} = \frac{R_1}{\frac{3}{2}p + 1}\,.$$

It is evident why we needed to work with two different cases since the recurrent sequence starts at a different index. We can let Mathematica calculate the general solution again

$$R_{bN} = \frac{2R_1}{\frac{2\sqrt{2p+1}}{\left(\frac{-1-p+\sqrt{2p+1}}{p^2}\right)^{-(N-1)/2}\left(-\frac{1+p+\sqrt{2p+1}}{p^2}\right)^{(N-1)/2} - 1} + \sqrt{2p+1} + 2p + 1}\,.$$

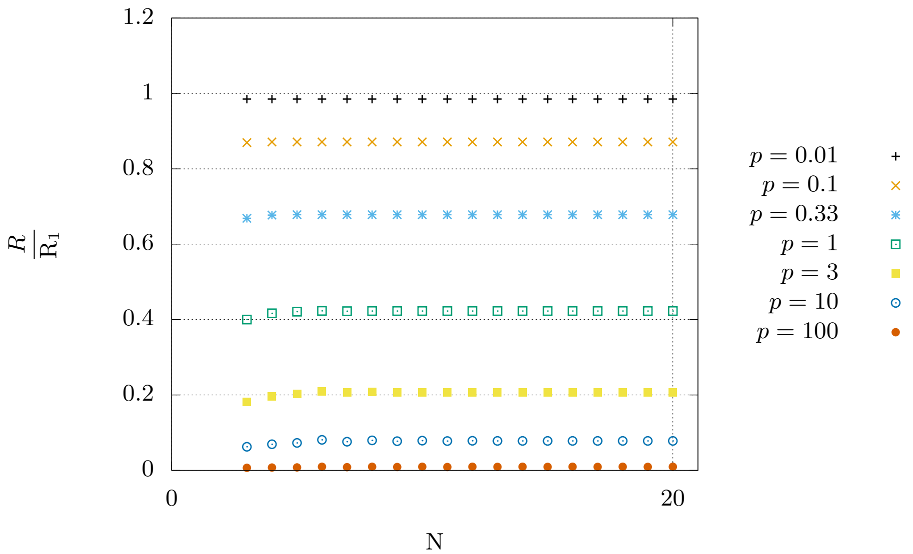

Fig. 3: Resistance in subtask b) depending on N and p.

Between opposite vertices in the main plane

The solution is quite similar if we are looking for the resistance between opposite vertices in the main plane. We turn the object in such a way that one of the vertices between which we are calculating the resistance is in the front, and the other vertex is in the back. Through these vertices, perpendicularly to the main plane, passes another plane of symmetry (let's call it the first plane of symmetry). Perpendicularly to this plane and the main plane, there is yet another plane of symmetry passing through the center of the N-gon (let's call it the second plane). This solution must also deal with two cases: when N is divisible by four and when it is not. That is because there is a difference in whether the second plane of symmetry crosses the N-gon in a vertex or cuts one of its sides in half.

N indivisible by four If N is not divisible by four, the second plane of symmetry crosses the N-gon in the midpoints of two of its sides (let's call them the border sides). In our calculations, we proceed from these sides, similar to the previous subtask. From one endpoint of a border side, current again flows to the other endpoint through a main vertex, and we do not need to consider a conductive contact in the main vertex.

We split the whole object between the front and back vertices into four branches. We have two branches, where a branch goes from the front vertex through the main vertices, and twice we have the side branches. The net resistance is formed by connecting these four branches in parallel. Once again, we construct a recurrence relation. Let's assume that the resistance $R_{c(N-4)}$ is known. Using it, we calculate the resistance of one branch by dividing its inverse by two and subtracting a branch (only one) that passes through a main vertex. Then we need to connect the two branches passing through the main vertices (each with resistance $2R_1$) in parallel and then $2R_2$ in series, which gives the resistance of a larger side branch. In the end, we again connect it in parallel with the other, identical branch and with a path through the main vertices. Finally, we can write it as an equation

$$\frac{1}{R_{cN}} = \frac{2}{2R_1} + \frac{2}{2R_2 + \left(\frac{2}{2R_1} + \frac{1}{2R_{c(N-4)}} - \frac{1}{2R_1}\right)^{-1}}\,.$$

Using the same substitutions as in the previous part, we transform this equation into

$$G_{cN} = 1 + p\frac{\left(1 + G_{c(N-4)}\right)}{1 + p + G_{c(N-4)}}\,.$$

The smallest N is $N = 6$. For this value, the resistance we are looking for is

$$R_{c6} = \left(2\frac{1}{2R_1} + 2\frac{1}{2R_2 + \frac{R_1R_2}{R_1+R_2}}\right)^{-1} = \frac{R_1}{1 + 2p\frac{1+p}{2p(1+p)+p}}\,,$$

which corresponds to

$$G_{c6} = 1 + 2p\frac{1+p}{3p+2}\,.$$

Notably, we would obtain this result by substituting

$$G_{c2} = G_{b2} = 2p + 1\,.$$

in the recurrence. Even with this initial condition, the general solution is unfortunately too ugly.

N divisible by four If N is divisible by four, then the second plane of symmetry crosses the N-gon in two of its vertices. Just like before, from these vertices, current will not flow to or from the main vertices due to symmetry, and we can ignore the wires between them. The recurrence is the same as in the previous part.

The smallest possible N here is $N = 4$, for which

$$R_{c4} = \left(\frac{1}{2R_2} + \frac{1}{2R_2} + \frac{1}{2R_1} + \frac{1}{2R_1}\right)^{-1} = \frac{R_1}{p+1}\,,$$

It corresponds to

$$G_{c4} = p + 1 \implies$$
$$\implies R_{cN} = \frac{R_1}{\frac{2\sqrt{2p+1}}{\left(\frac{-1-p+\sqrt{2p+1}}{p^2}\right)^{-N/4}\left(-\frac{1+p+\sqrt{2p+1}}{p^2}\right)^{N/4} - 1} + \sqrt{2p+1}}\,.$$

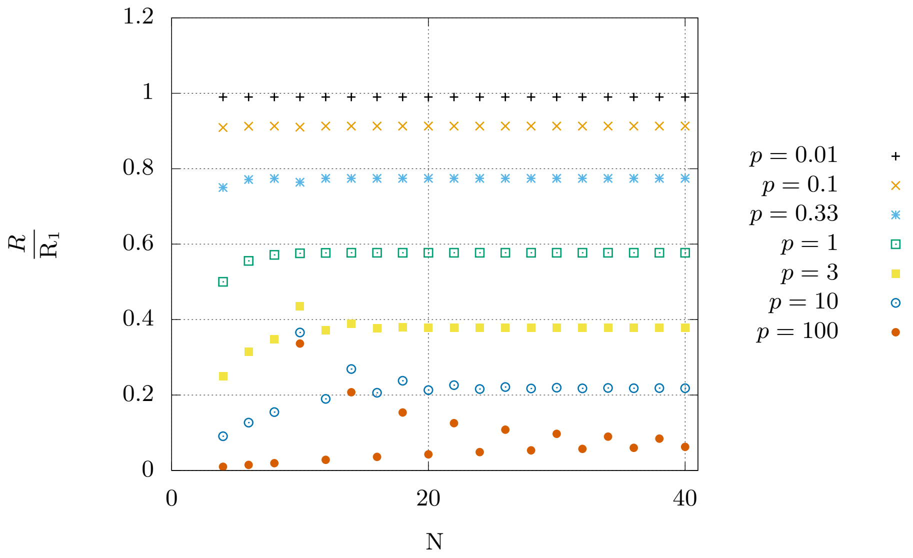

Fig. 4: Resistance in subtask c) depending on N and p.

We can see that for $N \to \infty$, the resistance value in both subtasks converges to some limit values. To calculate these limits, we assume the equality $R_{bN} = R_{bN-2}$ or $R_{cN} = R_{cN-4}$ and substitute into the recurrence relations. We get

$$R_{b\infty} = R_1 \frac{2}{2p + 1 + \sqrt{2p+1}} \quad \text{a} \quad R_{c\infty} = R_1 \frac{1}{\sqrt{1+2p}}\,.$$

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