Problem III.5 ... air under the water

10 points; průměr 3,97; řešilo 72 studentů

Assume a cylindrical glass of negligible mass, internal cross-sectional area S, and height h that is turned upside down and its open rim aligned with the water level in the reservoir. We start to push slowly downwards. What work will we perform if we move the jar with the air inside so that its base $d > 0$ is below the surface?

Bonus: Let us now consider a more realistic case. How much work must be performed to completely submerge a jar of the same dimensions but mass m to the bottom of a container with area A and initial water level in height H? Assume that the jar is completely submerged when it reaches the bottom.

Jarda would not like to visit Titanic...

As the problem is not as simple as it may seem at first glance, we will show several ways of solving it, which differ significantly in mathematical difficulty. Firstly, we will tackle the problem by employing a closed jar method. Next, we will translate the problem into determining the center of gravity. The third solution involves utilizing thermodynamic potential, specifically enthalpy. Lastly, we will compute the work directly from the applied force, which is the most straightforward approach but entails more mathematical complexity.

First, however, we will introduce the notation of the quantities. The bottom of the jar is at the top after turning it upside down. The top rim is at the bottom. The height of the inner volume of the jar (i.e., the jar's height) remains h, and the cross-sectional area of the interior is S. Once the jar is submerged, the height of the volume of air inside decreases, and we denote it by x (from the bottom of the jar downwards). The letter y stands for the distance of the bottom of the jar from the surface downwards, so in the initial position, $y = -h$. The density of water is ρ, the gravitational acceleration is g, the atmospheric pressure is p_{a}, and the pressure inside the jar is p (so $p = p_{\mathrm{a}}$ at the start). If the jar is submerged at $y = d$, we denote $x = x^*$. For $y = 0$, we denote the height of the air volume inside the jar by $x = x_0$.

Thought experiment with a closed jar

At the beginning of the problem, it is necessary to understand where our work will manifest itself. A jar in water is subject to a buoyant force $F_{\mathrm{v}} = V\rho g$, where V is the volume of air in the jar below the surface of the water, ρ is the density of the water, and g is the gravitational acceleration. This force acts upwards, so we need to do work to move the jar further under the water. Another force is the gravitational force of the air with invariable mass m in the jar, so its weight is $F_{\mathrm{g}} = mg$. However, the air density is about three orders of magnitude smaller than that of water at normal pressure and temperature, so we can neglect it in our solution.

We have to do more work compressing the air inside. The work required to compress the air at a pressure p by a small volume $\mathrm{d}V$ is $\mathrm{d}W = p\,\mathrm{d}V = pS\,\mathrm{d}x$, where we have written the differential of the volume $\mathrm{d}V$ as the product of the base S and the change in the height of the air in the jar $\mathrm{d}x$.

Since the volume of air in the jar changes with the depth of immersion due to compression, we solve the problem with a trick. The final state of the jar and the air in it does not depend on the process by which it is reached (as long as all parts of the process are reversible so that no energy is lost in the form of heat, for example).

Imagine that some piston, which will be allowed to move around in the jar, is also used to seal the bottom (normally the top) opening of the jar. For now, we leave the piston on the rim of the jar, i.e., at a distance h from its regular bottom, and immerse the jar in water so that its bottom (top base) is y below the surface. We have thus performed the work to overcome the buoyant force.

When we release the piston, its position changes because the water pressure is higher than the atmospheric pressure p_a left in the jar. The piston will move upwards until the pressures are equal. If we let go of the piston, it would compress the gas too much and could start oscillating. But if we let it go slowly, we could make some use of the work that the water does with its hydrostatic pressure. That would subtract that work from the work done by submerging the jar underwater.

Let's proceed in this way. The work required to overcome the buoyant force is equal to

$$W_\mathrm{v} = \int_{-h}^{0} (h+y)\, S\rho g \,\mathrm{d}y + \int_{0}^{d} hS\rho g \,\mathrm{d}y \,.$$

The second term describes a situation where the entire jar is submerged underwater. In this case, the applied buoyant force $F_\mathrm{v} = Sh\rho g$ is constant. The first term describes the situation where part of the jar is above water, and the buoyant force of the water is proportional only to the submerged part $h+y$. The coordinate y describes the distance of the bottom of the jar from the surface, so initially $y=-h$, and when the whole jar is submerged, $y=0$. At the end, $y=d$. The total work done is therefore

$$W_\mathrm{vz} = \frac{h^2}{2} S\rho g + hS\rho g d \,.$$

By doing this work, we have brought the jar to a depth d below the surface. Now, we are going to be slowly bringing the piston up. Let's denote x as the distance of the piston from the bottom of the jar (so at the beginning, $x=h$). The pressure of the water as a function of x is given by the sum of the atmospheric pressure p_a and the hydrostatic pressure

$$p_\mathrm{v} = (x+d)\,\rho g + p_\mathrm{a} \,.$$

Against the water pressure, the pressure p from inside the jar acts on the piston

$$p = p_\mathrm{a} \frac{h}{x} \,,$$

while its expression as a function of x was obtained from the isothermal process from the condition $p_1 V_1 = p_2 V_2$. The force acting on the piston is thus

$$F = S\,(p_\mathrm{v} - p) = S\left((x+d)\,\rho g + p_\mathrm{a} - p_\mathrm{a}\frac{h}{x}\right) \,.$$

The work performed by water is

$$W_\mathrm{v} = \int_{h}^{x^*} S\left((x+d)\,\rho g + p_\mathrm{a} - p_\mathrm{a}\frac{h}{x}\right)(-\,\mathrm{d}x) \,,$$

where x^* is the equilibrium position of the piston when the pressure of the compressed air inside the jar and the pressure of the water are equal. By integration, we get

$$\begin{aligned} W_\mathrm{v} &= S\left[\left(x^2/2 + dx\right)\rho g + p_\mathrm{a} x - p_\mathrm{a} h \ln x\right]_{x^*}^{h} \\ &= S\left(\left(\frac{h^2 - x^{*2}}{2} + d\,(h - x^*)\right)\rho g + p_\mathrm{a}\,(h - x^*) - p_\mathrm{a} h \ln\frac{h}{x^*}\right) \,. \end{aligned}$$

Now, the only task remaining is to determine the equilibrium position of the piston. We find it when the force acting on the piston is zero, i.e.

$$(x^* + d)\,\rho g + p_\mathrm{a} = p_\mathrm{a} \frac{h}{x^*}\,,$$

which is a quadratic equation for x with solution

$$x^{*2} \rho g + x^* d\rho g + x^* p_\mathrm{a} - p_\mathrm{a} h = 0 \quad \Rightarrow \quad x^* = \frac{-\left(d\rho g + p_\mathrm{a}\right) + \sqrt{\left(d\rho g + p_\mathrm{a}\right)^2 + 4 p_\mathrm{a} h \rho g}}{2\rho g}\,,$$

where we had to choose a positive root because the negative root would give negative x^*, which does not make sense. Substituting into the equation above, we get the work done by the water as

$$W_\mathrm{v} = S\left(\left(\frac{h^2 - x^{*2}}{2}\right)\rho g + \left(h - x^*\right)\left(d\rho g + p_\mathrm{a}\right) - p_\mathrm{a} h \ln \frac{h}{x^*}\right)\,.$$

Thus, the total work we had to do is

$$W_\mathrm{celk} = S\left(\frac{x^{*2}\rho g}{2} - h p_\mathrm{a} + x^* d\rho g + x^* p_\mathrm{a} + p_a h \ln \frac{h}{x^*}\right)\,.$$

Substituting from the quadratic equation the term $-x^{*2}\rho g = -p_\mathrm{a} h + x^* d\rho g + x^* p_\mathrm{a}$ we get

$$W_\mathrm{celk} = S\left(-\frac{x^{*2}\rho g}{2} + p_a h \ln \frac{h}{x^*}\right)\,,$$

which, after reaching the equilibrium position, finally leads to the result

$$\begin{aligned} W_\mathrm{celk} = {} & S\frac{-2\rho g p_\mathrm{a} h - \left(d\rho g + p_\mathrm{a}\right)^2 + \sqrt{\left(d\rho g + p_\mathrm{a}\right)^2 + 4 p_\mathrm{a} h \rho g}\left(d\rho g + p_\mathrm{a}\right)}{4\rho g} \\ & + S p_a h \ln \frac{2h\rho g}{-\left(d\rho g + p_\mathrm{a}\right) + \sqrt{\left(d\rho g + p_\mathrm{a}\right)^2 + 4 p_\mathrm{a} h \rho g}}\,. \end{aligned}$$

Thought experiment with grains

We can describe the whole process in a slightly different way. Imagine there are grains of sand on (continuous) shelves under the surface, all the way to depth h. The piston that we have used to close the bottom hole in the jar is hollow, allowing grains to enter it. The piston cannot move in the jar at first. We slide the top grains onto the piston, so the whole jar slightly sinks as we have increased the gravitational force. We push in the next grains, and the jar sinks again. We repeat this until all the grains are in the piston and the whole jar is submerged just below the surface so that the grains' weight offsets the buoyant force of the water. In this case, $m = V\rho$, where m is the mass of the grains and V is the volume of the jar.

The resultant of the forces acting on the jar with the piston and the grains is zero. We can move the jar to any depth without doing any work. When we reach the desired depth d (from the bottom of the jar to the surface), we fix the jar for a moment so that it cannot move (e.g., we strap it to the bottom). We slide all sand out of the piston and return it to its original location below the surface. It is here, by pulling out the sand, that we do the work needed to submerge the sealed jar in the water. The work performed corresponds to the change in the center of gravity of the sand in the gravitational field and is equal to

$$W_{\mathrm{vz}} = mg\left(d + h - \frac{h}{2}\right) = Sh\rho g\left(d + \frac{h}{2}\right) ,$$

which is, in its entirety, consistent with the expression above.

Now, we find more grains of sand on the shelf $d + h$ below the surface of the submerged jar. We slide enough of them into the piston to release the piston from its constraint but in a way that it does not move anywhere, i.e., so that the compressive force of the air inside and the gravitational force of the grains balance the pressure of the water. We slowly remove the grains and move them to the shelves next to the piston. Once we removed the grains, the piston moved up to balance the forces. The weight of the grains in the piston (depending on the volume of air in the jar) is thus the result of the equality of forces.

$$M(x)\,g + Sp_{\mathrm{a}}\frac{h}{x} = (x + d)\,\rho g S + p_{\mathrm{a}} S \quad \Rightarrow \quad M(x) = \frac{S}{g}\left((x + d)\,\rho g + p_{\mathrm{a}} - p_{\mathrm{a}}\frac{h}{x}\right) .$$

Let $M_0 = M(h) = S\,(h + d)\,\rho$ be the mass of the grains at the beginning of the piston's movement. We will need it in no time.

We remove the last grain from the piston at the equilibrium position x^*. The density of sand grains on the shelves next to the jar is then

$$\lambda(x) = \frac{\mathrm{d}m}{\mathrm{d}x} = \frac{S}{g}\left(\rho g + p_{\mathrm{a}}\frac{h}{x^2}\right) .$$

Thus, the water pressure, in addition to compressing the air, did the work of lifting the grains of sand. Therefore, we can only find their center of gravity to get the magnitude of this mechanical work done. We find the distance of the center of gravity from the bottom of the jar as

$$x_{\mathrm{T}} = \frac{1}{M_0}\int_{x^*}^{h} \lambda(x)\,x\,\mathrm{d}x = \frac{1}{(h + d)\,\rho g}\left(\rho g\frac{h^2 - {x^*}^2}{2} + p_{\mathrm{a}} h \ln\frac{h}{x^*}\right) .$$

The center of gravity of the grains has, therefore, risen from the shelf at a depth $h + d$ below the surface by

$$t = h - x_{\mathrm{T}} = \frac{1}{(h + d)\,\rho g}\left(hd\rho g + \rho g\frac{h^2}{2} + \rho g\frac{{x^*}^2}{2} - p_{\mathrm{a}} h \ln\frac{h}{x^*}\right) ,$$

so their potential energy has increased by

$$W_{\mathrm{v}} = M_0 g t = S\left(hd\rho g + \rho g\frac{h^2}{2} + \rho g\frac{{x^*}^2}{2} - p_{\mathrm{a}} h \ln\frac{h}{x^*}\right) .$$

We can utilize this gravitational potential energy of the grains, so we subtract it from the work required to lift the initial grains. The total work we had to do was, therefore

$$W_{\mathrm{celk}} = W_{\mathrm{vz}} - W_{\mathrm{v}} = S\left(-\rho g\frac{{x^*}^2}{2} + p_{\mathrm{a}} h \ln\frac{h}{x^*}\right) ,$$

which, after substituting in x^{ast^2}, leads again to the same result as in the previous section.

Solution using enthalpy

Enthalpy is the energy required to create a system of internal energy U and volume V in an environment with ambient pressure P. As a state function, it does not depend on the path, only on the current state of the system. We calculate it as

$$H = U + PV\,,$$

where the term PV is the energy required to create space in the environment for our system.

Suppose that within a distance $y \in [d, d + x^*]$ below the surface, we need to make room for the current air in the jar at pressure p_a and the volume Sh. The required volume that such an amount of air will have at depth d is Sx^*. Since the ambient pressure changes as the depth changes, we must integrate the PV term. The enthalpy of such a system is

$$H_\mathrm{f} = U + S\int_0^{x^*} \left(\left(d + x\right)\rho g + p_\mathrm{a}\right)\mathrm{d}x = U + S\left(\left(dx^* + \frac{{x^*}^2}{2}\right)\rho g + p_\mathrm{a} x^*\right)\,.$$

However, in the beginning, we had a system (air in a jar) with the initial enthalpy

$$H_\mathrm{i} = U + Sp_\mathrm{a}h\,.$$

The internal energy of the gas has not changed because it is an isothermal process. We have, therefore, performed work to change the enthalpy

$$H_\mathrm{f} - H_\mathrm{i} = S\left(\left(dx^* + \frac{{x^*}^2}{2}\right)\rho g + p_\mathrm{a} x^* - p_\mathrm{a}h\right)\,.$$

By substituting from the equation ${x^*}^2\rho g + dx^*\rho g + p_\mathrm{a}x^* = p_\mathrm{a}h$ we can modify the previous expression to

$$H_\mathrm{f} - H_\mathrm{i} = -\rho g S\frac{{x^*}^2}{2}\,.$$

Nevertheless, we have still done work to compress the gas, which is an isothermal process

$$W_\mathrm{s} = \int_h^{x^*} p\,\mathrm{d}x = \int_h^{x^*} \frac{p_\mathrm{a}h}{x}\,\mathrm{d}x = p_\mathrm{a}h\ln\frac{h}{x^*}\,.$$

This work escaped as heat from the jar walls to the surrounding water reservoir. In total, we performed the work

$$W_{celk} = -\rho g S\frac{{x^*}^2}{2} + p_\mathrm{a}h\ln\frac{h}{x^*}\,.$$

Direct calculation using forces

To submerge the jar, we need to apply a downward force. Our “helpers” in this task are the atmospheric pressure and the hydrostatic pressure exerted by the water above the jar, which submerges its bottom. Conversely, the air pressure inside the jar, which is equal to the sum of the hydrostatic and atmospheric pressure at the point where the water is pressing on the air in the jar, acts on the jar.

Once again, we divide the solution into two parts. The first is when part of the jar is still above water, and the second is when the whole jar is already under water. In the first case, the force that we have to overcome to submerge the jar further takes the form

$$F_1 = S\left(p - p_\mathrm{a}\right) ,$$

in the second case, hydrostatic interaction on the bottom of the jar is added

$$F_2 = S\left(p - \left(y\rho g + p_\mathrm{a}\right)\right) ,$$

where p is the pressure inside the jar. The equation for the equality of the pressures at the surface of the water in the jar in both cases is

$$\left(x + y\right)\rho g + p_\mathrm{a} = p = \frac{p_\mathrm{a} h}{x} ,$$

where $y \in [-h, 0]$ and x is the height of the air in the jar. We leave the notation x^* for the height x that the air will occupy when submerged below the surface d, i.e., as in the previous cases. Again, we find the height x_0 when the jar is completely submerged under water as $x_0 = (-p_\mathrm{a} + \sqrt{p_\mathrm{a}^2 + 4\rho g p_\mathrm{a} h})/(2\rho g)$.

When the force F_1 is applied to move the jar down the path $\mathrm{d}y$, the work done is

$$\mathrm{d}W_1 = F_1\,\mathrm{d}y = S\left(p - p_\mathrm{a}\right)\mathrm{d}y .$$

From the equality of the pressure in the jar and on the surface of the water in the jar, we can write the following for the differential of work

$$\mathrm{d}W_1 = S p_\mathrm{a}\left(\frac{h}{x} - 1\right)\mathrm{d}y .$$

The work done by this force serves both to push the jar deeper and to compress the air inside. However, we face the question of which variable to integrate by. By x or by y? In previous solutions, we saw that the result had a compact form for x^*, so we will integrate the variable x, which should be mathematically simpler. We will then try integrating y to show that the appropriate choice of variable is important in terms of mathematical difficulty. However, we must now express $\mathrm{d}y(x)$. We take the differentials of the variables in the equation for the pressures and get

$$\left(\mathrm{d}x + \mathrm{d}y\right)\rho g = -\frac{p_\mathrm{a} h}{x^2}\,\mathrm{d}x \quad\Rightarrow\quad \mathrm{d}y = -\frac{p_\mathrm{a} h + x^2\rho g}{x^2\rho g}\,\mathrm{d}x .$$

We see that the change in internal volume is negative as we move the jar deeper, which is the result we would expect.

The work required to submerge the whole jar under water is then given by the integral of the force F_1

$$\begin{aligned} W_1 &= -S p_\mathrm{a}\int_h^{x_0}\left(\frac{h}{x} - 1\right)\frac{p_\mathrm{a} h + x^2\rho g}{x^2\rho g}\,\mathrm{d}x = S p_\mathrm{a}\int_{x_0}^{h}\left(\frac{p_\mathrm{a} h^2}{x^3\rho g} - \frac{p_\mathrm{a} h}{x^2\rho g} + \frac{h}{x} - 1\right)\mathrm{d}x \\ &= S p_\mathrm{a}\left(-\frac{p_\mathrm{a}}{2\rho g}\left(1 - \frac{h^2}{x_0^2}\right) + \frac{p_\mathrm{a}}{\rho g}\left(1 - \frac{h}{x_0}\right) + h\ln\frac{h}{x_0} - h + x_0\right) \\ &= S p_\mathrm{a}\left(\frac{p_\mathrm{a}}{2\rho g}\left(1 - \frac{h}{x_0}\right)^2 + h\ln\frac{h}{x_0} - h + x_0\right) . \end{aligned}$$

As we move the jar further under the surface, we do work

$$\mathrm{d}W_2 = F\,\mathrm{d}y = S\left(p - (y\rho g + p_\mathrm{a})\right)\mathrm{d}y\,.$$

From the equality of the pressure in the jar and on the surface of the water in the jar, we can write for the differential of work

$$\mathrm{d}W_2 = Sx\rho g\,\mathrm{d}y\,.$$

Substituting for dy and integrating, we get

$$\begin{aligned} W_2 &= S\rho g \int_{x_0}^{x^*} x\,\mathrm{d}y \\ &= S\rho g \int_{x^*}^{x_0} x \frac{p_\mathrm{a}h + x^2\rho g}{x^2\rho g}\,\mathrm{d}x \\ &= S\int_{x^*}^{x_0} \frac{p_\mathrm{a}h}{x}\,\mathrm{d}x + S\rho g\int_{x^*}^{x_0} x\,\mathrm{d}x \\ &= Sp_\mathrm{a}h\ln\frac{x_0}{x^*} + S\rho g\frac{x_0^2 - x^{*2}}{2}\,. \end{aligned}$$

The total work done is the sum of $W_1 + W_2$, i.e.

$$W_\mathrm{celk} = Sp_\mathrm{a}h\ln\frac{h}{x^*} - S\rho g\frac{x^{*2}}{2} + \left(S\rho g\frac{x_0^2}{2} + Sp_\mathrm{a}\left(\frac{p_\mathrm{a}}{2\rho g}\left(1 - \frac{h}{x_0}\right)^2 - h + x_0\right)\right)\,,$$

where the whole parenthesis is zero.

$$\begin{aligned} &S\rho g\frac{x_0^2}{2} + Sp_\mathrm{a}\left(\frac{p_\mathrm{a}}{2\rho g}\left(1-\frac{h}{x_0}\right)^2 - h + x_0\right) \\ &\qquad = S\rho g\frac{x_0^2}{2} + Sp_\mathrm{a}\left(\left(\frac{x_0^2\rho g}{2p_\mathrm{a}}\right) - h + x_0\right) = S\rho g x_0^2 - hSp_\mathrm{a} + Sp_\mathrm{a}x_0 = 0\,, \end{aligned}$$

so we get our usual result

$$W_\mathrm{celk} = Sp_\mathrm{a}h\ln\frac{h}{x^*} - S\rho g\frac{x^{*2}}{2}\,.$$

What would happen if we chose y as the integration variable instead of x? We will indicate this by calculating W_2, which in this case would be

$$W_2 = S\rho g\int_0^d x\,\mathrm{d}y = S\int_0^d \frac{-(y\rho g + p_\mathrm{a}) + \sqrt{(y\rho g + p_\mathrm{a})^2 + 4p_\mathrm{a}h\rho g}}{2}\,\mathrm{d}y\,.$$

The integral of the first bracket in the integrand is equal to $-\left(y^2\rho g + 2yp_\mathrm{a}\right)S/4$. To calculate the square root, however, we would have to choose a substitution with hyperbolic sine

$$\frac{y\rho g + p_\mathrm{a}}{\sqrt{4p_\mathrm{a}h\rho g}} = \sinh\xi, \mathrm{d}y = \frac{\sqrt{4p_\mathrm{a}h\rho g}}{\rho g}\cosh\xi\,\mathrm{d}\xi\,,$$

which leads to

$$\begin{aligned}\frac{S}{2}\int\sqrt{(y\rho g+p_\mathrm{a})^2+4p_\mathrm{a}h\rho g}\,\mathrm{d}y&=S2p_\mathrm{a}h\int\sqrt{\sinh^2\xi+1}\cosh\xi\,\mathrm{d}\xi\\&=S2p_\mathrm{a}h\int\cosh^2\xi\,\mathrm{d}\xi=Sp_\mathrm{a}h\int(\cosh 2\xi+1)\,\mathrm{d}\xi\\&=\frac{Sp_\mathrm{a}h}{2}\sinh 2\xi+Sp_\mathrm{a}h\xi=Sp_\mathrm{a}h\sinh\xi\cosh\xi+Sp_\mathrm{a}h\xi\,.\end{aligned}$$

We leave the individual steps and the introduction to hyperbolic functions to the reader as homework. By backtracking from substitution and considering integration limits, we get the work W_2 as

$$\begin{aligned}W_2=\frac{S}{4\rho g}&\left((d\rho g+p_\mathrm{a})\sqrt{(d\rho g+p_\mathrm{a})^2+4p_\mathrm{a}h\rho g}-p_\mathrm{a}\sqrt{p_\mathrm{a}^2+4p_\mathrm{a}h\rho g}\right)\\+Sp_\mathrm{a}h&\left(\operatorname{argsinh}\frac{d\rho g+p_\mathrm{a}}{\sqrt{4p_\mathrm{a}h\rho g}}-\operatorname{argsinh}\frac{p_\mathrm{a}}{\sqrt{4p_\mathrm{a}h\rho g}}\right)-\frac{S}{4}\left(d^2\rho g+2dp_\mathrm{a}\right)\,.\end{aligned}$$

Further adjustments would ensure we get the same result as in the previous section. However, the path to it is mathematically much more complicated.

Solution of bonus

After we have thoroughly discussed the solution of the basic part of the problem, the solution of the bonus is almost trivial. Let's consider that at the original level H, we make a hole in the wall of the container. Thus, as the jar is gradually submerged, the water is held at a constant height H, and the situation will be the same as in the previous section.

Now, let us suppose that the water that flows out of the container does not fall, but we hold it at the height H. Once the jar is submerged to the bottom, the amount of water is proportional to the volume of air in the jar

$$m_\mathrm{v}=\rho Sx^*\,.$$

However, we must return the water to the container to achieve the desired setup. Since the container is full up to a height of H, we need to raise the center of gravity of the displaced water. For the height of the water, after we put it back into the jar, the conservation of volume will be $A\Delta H=x^*S$, so we need to do work to raise its center of gravity by $\Delta H/2$

$$W_\mathrm{zv}=\frac{m_\mathrm{v}g\Delta H}{2}=\frac{\rho S^2{x^*}^2g}{2A}\,.$$

By submerging the jar of mass m, we gained work

$$W_\mathrm{pon}=mgH\,,$$

as its center of gravity has been reduced by H. The total work required to immerse the jar to the bottom of the container is thus

$$W_\mathrm{bonus}=S\left(-\rho g\frac{{x^*}^2}{2}+p_\mathrm{a}h\ln\frac{h}{x^*}\right)+\frac{\rho S^2{x^*}^2g}{2A}-mgH\,.$$

After substituting $d = H - h$ into $x^* = \left(-\left(d\rho g + p_\mathrm{a}\right) + \sqrt{\left(d\rho g + p_\mathrm{a}\right)^2 + 4p_\mathrm{a}h\rho g}\right)/(2\rho g)$ and then into the previous equation, we get

$$W_\mathrm{bonus} = \frac{S}{8\rho g}\left(\frac{S}{A} - 1\right)\left(-\left(\left(H-h\right)\rho g + p_\mathrm{a}\right) + \sqrt{\left(\left(H-h\right)\rho g + p_\mathrm{a}\right)^2 + 4p_\mathrm{a}h\rho g}\right)^2$$
$$+ p_\mathrm{a}Sh\ln\frac{2\rho g h}{-\left(\left(H-h\right)\rho g + p_\mathrm{a}\right) + \sqrt{\left(\left(H-h\right)\rho g + p_\mathrm{a}\right)^2 + 4p_\mathrm{a}h\rho g}} - mgH\,.$$

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FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Media Communications and PR Office and is supported by Institute of Theoretical Physics of MFF UK, his employees and The Union of Czech Mathematicians and Physicists. The realization of this project was supported by Ministry of Education, Youth and Sports of the Czech Republic.