Problem I.4 ... mountain transport

8 points; (chybí statistiky)

There is a town on the slope of a hill whose shape is a cone with apex angle $\alpha = 90^\circ$. On the other side of the cone, right opposite to the town in the same altitude, lies a train station. Mayor of the town decided to build a road to the station. They can either drill a tunnel or build a road on the surface of the hill. What is the maximum ratio of per-kilometer prices for the tunnel and for the surface road, so that building a tunnel is cheaper? The road can be built anywhere on the hill.

Matěj builds Semmeringbahn.

The mayor has several options for how to build the road. He can build a tunnel from the town directly to the train station (option A) or run the entire route along the mountain's surface (option B). They can also run part of the road from the town on the surface, through a tunnel, and then on the surface to the railway station (option C).

Option A

Let us mark the distance of the town from the top of the mountain l. If α is the angle town–peak–station, then the spatial distance s_A of the town from the station is

$$s_{\mathrm{A}} = 2l \sin\frac{\alpha}{2}\,,$$

which is also the length of the tunnel from the town to the station.

Option B

Option B is more complex, as it may not be apparent now which way the path should lead. In option A, we have led the tunnel along the shortest possible path in space – the line segment. Now it would seem that we have to lead the path along a curved surface, which is more challenging.[1] In reality, however, the surface of the cone is not curved as it might seem. In fact, we can expand it and decompose the cone shell into a flat 2D surface.[2] After this expansion, we can now easily connect the town to the railroad using a line segment and calculate its length.

The circumference of the base of the cone is $o = \pi s_{\mathrm{A}}$ and the mouth of the tunnel divides it in half. The shell of the cone is a circular section with central angle $\varphi = o/l$. The path on the surface is thus formed by the base of an isosceles triangle with arms of length l and angle $\varphi/2$ between them. The length of this path is therefore

$$s_{\mathrm{B}} = 2l \sin\frac{\varphi}{4} = 2l \sin\left(\frac{1}{2}\pi \sin\frac{\alpha}{2}\right)\,,$$

which is the shortest possible road between the town and the railway along the surface of the mountain.

Option C

The third option can be shown to be suboptimal without the need for calculations. The symmetry of the problem implies that the path from the town to the beginning of the tunnel must be plane symmetric to the path from the end of the tunnel to the station. That's the only way

[1] In the general case, we could resort to differential geometry and use the geodesic equation. *Geodesics* are in fact the shortest joints of two points in curved space(time).

[2] We can say that the surface of the cone does not have *internal* curvature, but only *external* curvature, which is determined by how the "straight" 2D shell is embedded in 3D space.

we'll be able to bring the ends of the tunnels closer together so that we can dig the shortest tunnel possible. Suppose that the most efficient solution to our problem would be option C. That is, build a road from the town on the surface to some point, tunnel from there to a point of symmetry, and from there lead the way to the railway.

We now show that in the optimal solution the tunnel cannot pass through the axis of the mountain. Let's look at the ratio of the length of the tunnel to the length of the imaginary surface path saved by the tunnel. In the case of a tunnel passing off the axis of the mountain, this ratio is certainly greater than when the tunnel passes through the axis. This holds because the imaginary surface path takes a relatively smaller detour (relative to the length of the tunnel) when the tunnel is closer to the surface. There is thus a lower cost saving and therefore it is never worth building such a shorter off-axis tunnel.

If the tunnel went through the axis of the mountain, the road would be constructed in a rather awkward way – the road would lead from the town and the station directly towards the top of the mountain and the tunnel would be built horizontally at a higher elevation. If this were to be the optimal design, it means that it is not worthwhile to run a surface road between these higher ends of the tunnel (analogous to the road in case B). However, since the ratio of the length of A to B is independent of the height of the mountain, the strategy with the tunnel at a higher elevation does not make economic sense, because we are just adding two unpleasant sections of the path.

By this contradiction, we have shown that we do not have to consider option C.

Conclusion

The critical value of ratio of per-kilometer prices for the tunnel and for the surface road is the ratio of the length of the surface road to the length of the tunnel

$$\frac{s_{\mathrm{B}}}{s_{\mathrm{A}}} = \frac{\sin\left(\frac{1}{2}\pi\sin\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}} \doteq 1.267\,,$$

If the price ratio is less than the critical value, it is worth digging the tunnel. If the ratio is greater, it pays off to lead the route around the mountain. In practice, we can expect the price ratio to be an order of magnitude larger. Still, it is sometimes worth digging tunnels because real mountains are not conical and it is often impossible to build a road around them.

Note that if we did not realize that the road does not necessarily have to run horizontally, the critical ratio would be larger, namely $\pi/2 \doteq 1.571$.

Matěj Mezera
m.mezera@fykos.org

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