Problem III.5 ... guitar

10 points; průměr 6,32; řešilo 66 studentů

Assume you have a guitar that is perfectly tuned at room temperature. By how many semitones (in tempered tuning) will the individual strings be out of tune if we move to a campfire, where it is cooler by $10\,^{\circ}\mathrm{C}$*? Will the guitar still sound in tune? The distance between the string attachment points is* $d = 65\,\mathrm{cm}$*. The strings have a density* $\rho = 8\,900\,\mathrm{kg{\cdot}m^{-3}}$*, a Young's modulus of elasticity* $E = 210\,\mathrm{GPa}$ *and a thermal expansion coefficient* $\alpha = 17\cdot 10^{-6}\,\mathrm{K^{-1}}$*.*

Honza's guitar is out of tune again.

string frequency

The string frequency is calculated from the equation

$$f = \frac{c}{\lambda} = \frac{\sqrt{\frac{F}{\mu}}}{2d}\,,$$

where μ is the length density of the string. Let us consider where the force that stretches the string comes from. The rest length of the stretched segment of string L_0 is less than d. Therefore, the string must be strained by a force of magnitude F to extend it to length d. However, if we move to a colder environment, the string tends to shorten but is still firmly attached. Therefore, a bigger force must be applied to the string. Hooke's law says

$$\sigma = E\varepsilon\,,$$
$$\frac{F}{S} = E\varepsilon\,,$$
$$\varepsilon = \frac{F}{SE}\,,$$

where σ is the mechanical stress, ε is the strain of the string and S is the cross-sectional area of the string. The total length of the stretched string will always be d and is related to L_0 as follows

$$d = L_0 + \varepsilon L_0 = L_0\left(1+\varepsilon\right)\,.$$

From the preceding discussion, it appears that after including the thermal expansion of the string, the equation for its length is

$$d = L_0\left(1+\alpha T\right)\left(1+\varepsilon\right) = L_0\left(1+\alpha T\right)\left(1+\frac{F}{SE}\right)\,.$$

From this equation, we can express the force F and plug it into the first equation

$$F = SE\left(\frac{d}{L_0\left(1+\alpha T\right)} - 1\right)\,.$$

Now we can express the frequency

$$f = \frac{\sqrt{\frac{SE\left(\frac{d}{L_0(1+\alpha T)}-1\right)}{\mu}}}{2d}\,.$$

We modify the expression $\frac{S}{\mu}$

$$\frac{S}{\mu} = \frac{S}{\frac{m}{d}} = \frac{1}{\frac{m}{dS}} = \frac{1}{\rho}$$

and plug into the formula for frequency

$$f = \frac{\sqrt{\frac{E\left(\frac{d}{L_0(1+\alpha T)}-1\right)}{\rho}}}{2d} \,.$$

If we put the temperature difference $T = 0\,^\circ\mathrm{C}$, we get the frequency f_0 that the string had indoors

$$f_0 = \frac{\sqrt{\frac{E\left(\frac{d}{L_0}-1\right)}{\rho}}}{2d} \,.$$

$$L_0 = \left(\frac{d}{\frac{4d^2 f_0^2 \rho}{E}+1}\right) \,.$$

From the formula for f_0, we obtained the formula for the unknown L_0. We can plug this into the formula for f and modify the resulting expression. For the reasons that will be explained later, we express the ratio of the frequencies: $\frac{f}{f_0}$

$$f = \sqrt{\frac{f_0^2 - \frac{E\alpha T}{4d^2\rho}}{1+\alpha T}} \,,$$

$$\frac{f}{f_0} = \sqrt{\frac{1 - \frac{E\alpha T}{4d^2\rho f_0^2}}{1+\alpha T}} \,.$$

Music Theory

In music theory, a tone an octave higher has twice the frequency of the original tone. Furthermore, the ratio of the frequencies of two notes that are a semitone apart is always constant (let us denote the constant K). Since an octave is made up of twelve semitones, the following is true

$$f_{\text{higher}} = 2f_0 \,,$$
$$f_{\text{higher}} = K^{12} f_0 \,,$$
$$K^{12} = 2 \,,$$
$$K = 2^{\frac{1}{12}} \,.$$

We can calculate by how many semitones the string is retuned

$$\frac{f}{f_0} = \left(2^{\frac{1}{12}}\right)^n ,$$

$$\left(2^{\frac{1}{12}}\right)^n = \sqrt{\frac{1 - \frac{E\alpha T}{4d^2\rho f_0^2}}{1 + \alpha T}} ,$$

$$n = 6 \log_2 \left(\frac{1 - \frac{E\alpha T}{4d^2\rho f_0^2}}{1 + \alpha T}\right) ,$$

where n is the number of semitones by which a string tuned to the frequency f_0 is re-tuned. The calculated values can be neatly summarized in a table.

Tab. 1: Changes in the frequency of the strings

string	$\frac{f_0}{\mathrm{Hz}}$	$\frac{f}{\mathrm{Hz}}$	n
E	330	334	0.19
H	247	252	0.33
G	196	202	0.52
D	146	154	0.92
A	110	120	1.55
e	82	95	2.62

Notice that strings tuned to higher frequencies do not get out of tune much due to temperature changes, while strings at lower frequencies can get out of tune by a whole tone under usual temperature differences. Since this change is uneven, meaning that all the strings do not retune by the same amount (e.g. one tone), the instrument will become out of tune.

We should also mention that during the calculation we neglected the fact that the cross-sectional area is reduced due to stretching. If we wanted to account for this effect, we would have to use an additional material constant called Poisson's ratio, which describes how the material shrinks in a direction perpendicular to the direction of stretching. If we included this effect, the calculated values would not differ up to the fourth or fifth significant digits. We can therefore disregard it with a cool head.

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