Problem II.3 ... decaying planet

5 points; (chybí statistiky)

Consider a planet with the same total mass and radius as the Earth. How much uranium ^{238}U would it have to contain, so that its surface temperature is $15\,^{\circ}\mathrm{C}$, assuming it is not lit by any nearby star.

Jarda got burned on the sun

Radiated power, according to the Stefan-Boltzmann law

$$P = 4\pi R^2 \sigma T^4$$

must be supplied by nuclear decay in equilibrium. In our case, the radius of the planet is the same as the Earth's radius, so $R = R_{\oplus}$, $\sigma = 5.67 \cdot 10^{-8}\,\mathrm{W \cdot m^{-2} \cdot K^{-4}}$ is the Stefan-Boltzmann constant and $T = 288\,\mathrm{K}$ is the thermodynamic temperature of the planet.

Ultimately, an energy of $E = 51.8\,\mathrm{MeV}$ belongs to each nucleus $^{238}\mathrm{U}$.[1] We must account for the entire decay chain, not just the decay of uranium. The half-lives of other elements are several orders of magnitude shorter than the half-life of $^{238}\mathrm{U}$. Which is $4.47 \cdot 10^{9}\,\mathrm{y}$. We can, therefore, assume that the rate of formation of all isotopes in the chain does not vary on scales of the order of millions of years. The nuclei that are now decaying could have been formed tens of thousands of years ago (e.g., nuclei with a half-life of about $250\,000\,\mathrm{y}$), but there are still just as many of them, so we can suppose that they decay instantaneously. Thus, we further consider that uranium decays directly with energy E. Essential is the conversion from the unit eV to J, which is an SI unit. We get $51.8\,\mathrm{MeV} = 8.3 \cdot 10^{-12}\,\mathrm{J}$.

The uranium activity is $A = N \cdot \ln 2 / T_{1/2}$, where N is the number of atoms in the planet's crust. Because of the long half-life, we can assume the number of nuclei is constant. The power is equal to the product of the activity and the energy released per one decay, so the number of nuclei in the planet must be:

$$N = \frac{4\pi}{\ln 2} \frac{T_{1/2} R^2 \sigma T^4}{E}\,.$$

The product of the number of nuclei N then gives the total mass of uranium required, and the mass of one nucleus $A_{\mathrm{U}} m_{\mathrm{u}}$, where $A_{\mathrm{U}} = 238$ is the relative atomic mass of uranium and $m_{\mathrm{u}} = 1.66 \cdot 10^{-27}\,\mathrm{kg}$ atomic mass unit, so

$$m = A_{\mathrm{U}} m_{\mathrm{u}} \frac{4\pi}{\ln 2} \frac{T_{1/2} R^2 \sigma T^4}{E} = 1.94 \cdot 10^{21}\,\mathrm{kg}\,.$$

Comparing it with the Earth, which has a mass of $6 \cdot 10^{24}\,\mathrm{kg}$, it is not even a thousandth of its mass. On the other hand, the mass of uranium in the Earth's crust is about five orders of magnitude smaller, so an Earth without a sun would be significantly colder than it is.

Jaroslav Herman
jardah@fykos.org

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Media Communications and PR Office and is supported by Institute of Theoretical Physics of MFF UK, his employees and The Union of Czech Mathematicians and Physicists. The realization of this project was supported by Ministry of Education, Youth and Sports of the Czech Republic.

[1] We can find it here, for example: https://en.wikipedia.org/wiki/Uranium-238.